Quantization of Spherically Symmetric Solution of *SU*(3) Yang–Mills Theory

V. Dzhunushaliev[1] and D. Singleton[2]

A recent investigation of the *SU*(3) Yang–Mills field equations found several classical solutions which exhibited a type of confinement due to gauge fields which increased without bound as $r \to \infty$. This increase of the gauge fields gave these solutions an infinite field energy, raising questions about their physical significance. In this paper we apply some ideas of Heisenberg about the quantization of strongly interacting nonlinear fields to this classical solution and find that at large r this quantization procedure softens the unphysical behavior of the classical solution, while the interesting short-distance behavior is maintained. This quantization procedure may provide a general method for approximating the quantum corrections to certain classical field configurations.

1. INTRODUCTION

Recently several classical solutions to *SU*(3) Yang-Mills theory have been discussed which possessed either spherical or cylindrical symmetry.[1] These solutions had gauge fields which tended toward ∞ at large distances, leading to a type of confining behavior if one considered these solutions as background fields in which some test particle moved. These increasing gauge fields also led these solutions to the undesired property of having infinite field energy. One way in which these classical field configurations might nevertheless have some physical importance is if the quantization of these solutions reduced or eliminated the bad long-distance behavior. While perturbative quantization techniques work well for weakly interacting field theories such as QED (or QCD in the high-energy limit), they are not useful

[1] Theoretical Physics Department, Kyrgyz State National University, 720024, Bishkek, Kyrgyzstan; e-mail: dzhun@freenet. bishkek.su.
[2] Department of Physics, CSU Fresno, Fresno, California 93740-8031; das3y@maxwell. phys.csufresno.edu .

when dealing with strongly interacting field theories. In ref. 2 we applied some ideas of Heisenberg's concerning the quantization of strongly interacting nonlinear fields[3] to the cylindrical solution discussed in ref. 1 and found that under certain assumptions the bad long-distance behavior of this solution was eliminated. Here we apply the same procedure to the spherically symmetric solution and show that again the bad long-distance behavior is eliminated. In addition to the specific benefit that the Heisenberg quantization method gives to the infinite-energy solutions discussed here and in ref. 2, it may provide some general procedure for approximating the quantum corrections to certain classical field configurations.

2. SPHERICALLY SYMMETRIC ANSATZ

We will briefly review the derivation and discuss some aspects of the spherically symmetric solution. We take the ansatz for the $SU(3)$ gauge field as in refs. 4–6:

$$A_0 = \frac{2\varphi(r)}{\mathbf{i}r^2} (\lambda^2 x - \lambda^5 y + \lambda^7 z) + \frac{1}{2} \lambda^a (\lambda_{ij}^a + \lambda_{ji}^a) \frac{x^i x^j}{r^2} w(r) \tag{1a}$$

$$A_i^a = (\lambda_{ij}^a - \lambda_{ji}^a) \frac{x^j}{\mathbf{i}r^2} (f(r) - 1) + \lambda_{jk}^a (\epsilon_{ilj} x^k + \epsilon_{ilk} x^j) \frac{x^l}{r^3} v(r) \tag{1b}$$

where λ^a are the Gell-Mann matrices; $a = 1, 2, \ldots, 8$ is a color index; the Latin indices $i, j, k, l = 1, 2, 3$ are the space indices; $\mathbf{i}^2 = -1$; and r, θ, ϕ are the usual spherical coordinates. Substituting the ansatz of Eqs. (1) (with $f = \phi = 0$) into the Yang–Mills equations

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{a\mu}_\nu) + f^{abc} F^{b\mu}_\nu A^c_\mu = 0 \tag{2}$$

yields the following complex set of coupled nonlinear differential equations[4]:

$$r^2 f'' = f^3 - f + 7fv^2 + 2vw\varphi - f(w^2 + \varphi^2) \tag{3a}$$

$$r^2 v'' = v^3 - v + 7vf^2 + 2fw\varphi - v(w^2 + \varphi^2) \tag{3b}$$

$$r^2 w'' = 6w(f^2 + v^2) - 12fv\varphi \tag{3c}$$

$$r^2 \varphi'' = 2\varphi(f^2 + v^2) - 4fvw \tag{3d}$$

For the solution with increasing gauge fields we specialize by taking $f = \varphi = 0$ (the case where $v = w = 0$ is similar) and find the following set of nonlinear coupled equations:

$$r^2 v'' = v^3 - v - vw^2 \tag{4a}$$

$$r^2 w'' = 6wv^2 \tag{4b}$$

Near $r = 0$ we take the series expansions for v and w as

$$v = 1 + v_2 \frac{r^2}{2!} + \dots \tag{5a}$$

$$w = w_3 \frac{r^3}{3!} + \dots \tag{5b}$$

where v_2, w_3 are constants which determine the initial conditions on v and w. In the asymptotic limit $r \to \infty$ the form of the solutions to Eqs. (4) approaches

$$v \approx A\sin(x^\alpha + \phi_0) \tag{6a}$$

$$w \approx \pm\left[\alpha x^\alpha + \frac{\alpha - 1}{4} \frac{\cos(2x^\alpha + 2\phi_0)}{x^\alpha} \right] \tag{6b}$$

$$3A^2 = \alpha(\alpha - 1) \tag{6c}$$

where $x = r/r_0$ is a dimensionless radius and r_0, ϕ_0, and A are constants. The second, strongly oscillating term in $w(r)$ is kept since it contributes to the asymptotic behavior of w''. We have not found an analytical solution for the system (4), but it is straightforward to solve these equations numerically with any standard differential equation package such as that available in *Mathematica*.[7] Figure 1 shows a representative solution to Eqs. (4). The exponent of the power law increase of w (which is represented by α in the asymptotic expressions) depends on the initial conditions, which are determined by the constants v_2, w_3. Generally the exponent α decreases from a value in the range $2-3$ to a value in the range $1.2-1.8$ for a wide range of initial conditions. This behavior can be seen in the $\log(w)$ versus $\log(x)$ plot in Fig. 2. Although these classical gauge fields weaken slightly as r increases, they still diverge as $r \to \infty$. Due to this feature of the ansatz function w the time part of the gauge field grows without bound as $r \to \infty$, leading to both a classical type of confinement (a test particle placed in the background field of this solution would not be able to escape to ∞) and an undesired infinite field energy for this solution. Various phenomenological studies of quarkonia bound states use such increasing potentials to study the spectrum of the bound state,[8] although usually the potential is taken to increase linearly. It should be mentioned that the asymptotic form of the classical solution given in (6) is expected to be altered by the quantum corrections. The classical short-distance behavior as given in Fig. 1 should be roughly correct, since the pure gauge *SU*(3) theory that we are considering is asymptotically free.
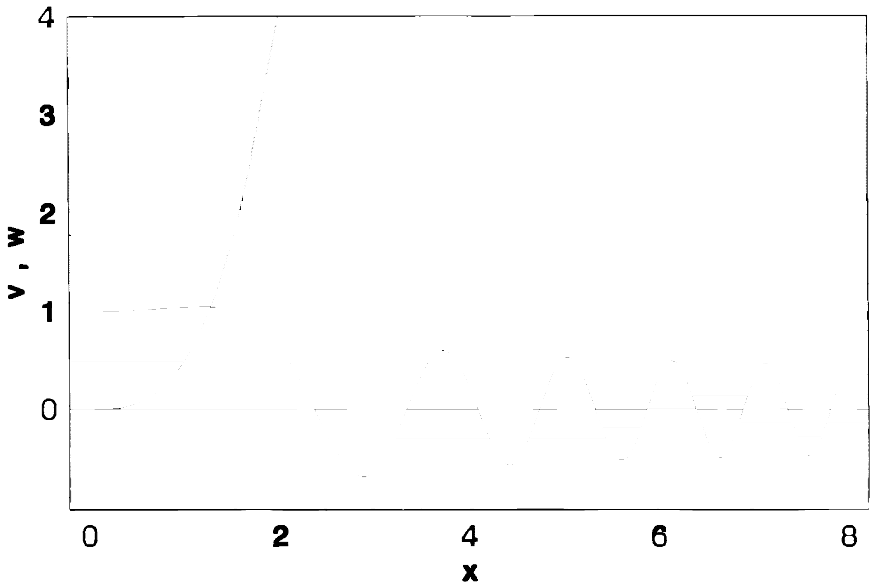
Fig. 1. The $w(x)$ confining function and the $v(x)$ oscillating function of the $SU(3)$ bunker solution. The initial conditions for this particular solution are $v_2 = 0.1$, $w_3 = 2.0$, and $x_i = 0.001$.
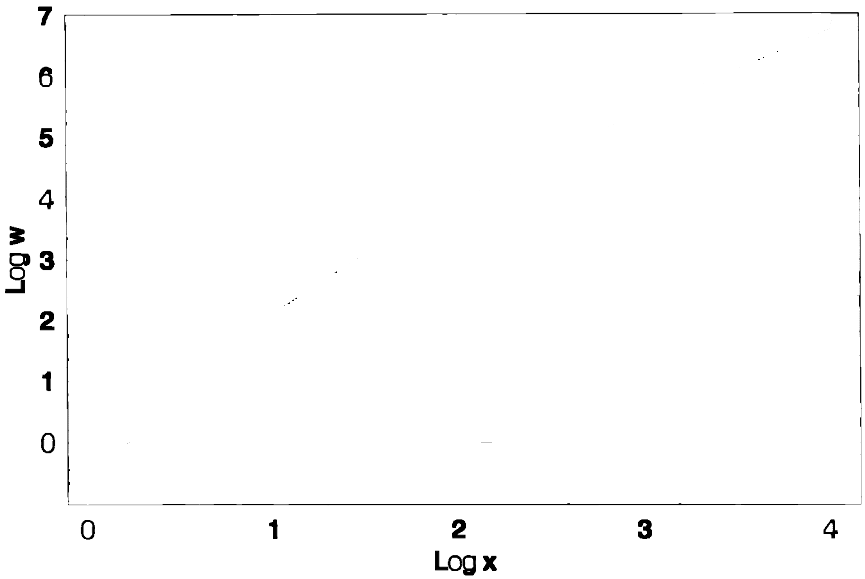

Fig. 2. A plot of $\log(w)$ versus $\log(x)$ of the solution from Fig. 1 showing the different power law behavior in the small x and large x regions.

This "bunker" solution has "magnetic" and "electric" fields associated with it. Using the ansatz for A_μ from Eq. (1), we find these "magnetic" and "electric" fields have the following proportionalities:

$$H_r^a \propto \frac{v^2 - 1}{r^2}, \qquad H_\varphi^a \propto v', \qquad H_\theta^a \propto v' \qquad (7a)$$

$$E_r^a \propto \frac{rw' - w}{r^2}, \qquad E_\varphi^a \propto \frac{vw}{r}, \qquad E_\theta^a \propto \frac{vw}{r} \qquad (7b)$$

where for E_r^a, H_θ^a, and H_φ^a the color index is $a = 1, 3, 4, 6, 8$ and for H_r^a, E_θ^a, and E_φ^a, $a = 2, 5, 7$.

The asymptotic behavior of H_φ^a, H_θ^a and E_φ^a, E_θ^a is dominated by the strongly oscillating function $v(r)$. It may be postulated that quantum corrections to this strongly oscillating solution tend to smooth it out, so that it would not play a significant role in the large-r limit. From Eqs. (7) and the asymptotic form of $v(r)$, $w(r)$ the radial components of the "magnetic" and "electric" have the following asymptotic behavior

$$H_r^a \propto \frac{1}{r^2}, \qquad E_r^a \propto \frac{1}{r^{2-\alpha}} \qquad (8)$$

where the strongly oscillating portion of H_r^a is assumed not to contribute in the limit of large r due to smoothing by quantum corrections. The radial "electric" field falls off slower than $1/r^2$ (since $\alpha > 1$) indicating the presence of a confining potential. The $1/r^2$ falloff of H_r^a indicates that this solution carries a "magnetic" charge. This was also true for the simple solutions discussed in refs. 4 and 5. This leads to the result that if a test is placed in the background field of the bunker solution, the composite system will have unusual spin properties (i.e., if the test particle is a boson, the system will behave as a fermion, and if the test particle is a fermion, the system will behave as a boson). This is the spin from isospin mechanism.[9]

By examining the classical *SU*(3) field equations. (4) we have found field configurations which lead to a classical confining behavior and which have some similarities with certain phenomenological models used to study heavy quark bound states. The most significant drawback of the present solution is that it has infinite field energy. The asymptotic form of the energy density goes as

$$\mathcal{E} \propto 4\frac{v'^2}{r^2} + \frac{2}{3}\left(\frac{w'}{r} - \frac{w}{r^2}\right)^2 + 4\frac{v^2 w^2}{r^4} + \frac{2}{r^4}(v^2 - 1)^2$$

$$\approx \frac{2}{3}\frac{\alpha^2(\alpha - 1)(3\alpha - 1)}{x^{4-2\alpha}} \qquad (9)$$

Since we found $\alpha > 1$, this energy density will yield an infinite field energy when integrated over all space. This can be compared with the finite field energy monopole[10] and dyon solutions.[11,12]

3. QUANTIZATION OF THE "BUNKER" SOLUTION

Although the classical confining behavior of this "bunker" solution may seem interesting due to its similarity with certain phenomenological potentials, the infinite field energy discussed at the end of the previous section strongly argues against the physical importance of this solution. One possible escape from this conclusion is if quantum effects weakened or removed the bad long distance behavior of these solutions. However, strongly interacting nonlinear theories are notoriously hard to quantize. In order to take into account the quantum effects on the bunker solution we employ a method used by Heisenberg[3] in attempts to quantize the nonlinear Dirac equation. By applying the dynamical equation of motion (in Heisenberg's case the nonlinear Dirac equation) to an n-point Green's function G_n, one would arrive at an equation relating G_n to higher order Green's functions (G_{n+1}, for example). Then applying the dynamical equations of motion to the higher Green's functions, one would get equations relating these higher Green's functions to even larger order Green's functions. Continuing in this way, one arrives at an infinite set of differential equations relating Green's functions of all orders. To handle this, Heisenberg employed the Tamm–Dankoff method whereby he only considered Green's functions up to some order, thus cutting off the infinite set of equations. Here we will employ a similar method to the "bunker" solution in terms of the ansatz functions v and w. Previously[2] we used this method on an infinite-energy, stringlike classical solution to the $SU(3)$ equations. For more details on the application of the Heisenberg method to such classical solutions we refer the reader to that article.

In order to use Heisenberg's quantization method on the nonlinear equations we make the following assumptions:

1. The degrees of freedom relevant for studying the "bunker" solution (both classically and also quantum mechanically) are given entirely by the two ansatz functions w, v. No other degrees of freedom arise through the quantization process.

2. From Eq. (6b) and Fig. 1, w is a smoothly varying function for large x, while v is strongly oscillating. Thus we take $w(x)$ to be almost a classical degree of freedom, while $v(x)$ is treated as a fully quantum mechanical degree of freedom. One might think that in this way only the behavior of v would change while w stayed the same. However, since w and v are coupled via the equations of motion, we find that both functions are modified.

To begin we replace the ansatz functions by operators $\hat{w}(x)$, $\hat{v}(x)$:

$$x^2\hat{v}'' = \hat{v}^3 - \hat{v} - \hat{v}\hat{w}^2 \tag{10a}$$

$$x^2\hat{w}'' = 6\hat{w}\hat{v}^2 \tag{10b}$$

where the prime denotes a derivative with respect to x. Taking into account assumption 2, we let $\hat{w} \to w$ become just a classical function again, and replace $\hat{v}^2$ in Eq. (10b) by its expectation value to arrive at

$$x^2\hat{v}'' = \hat{v}^3 - \hat{v} - \hat{v}w^2 \tag{11a}$$

$$x^2w'' = 6w\langle v^2 \rangle \tag{11b}$$

where the expectation value $\langle \hat{v}^2 \rangle$ is taken with respect to fluctuations of v around the classical solution of Eq. (6) or Fig. 1 (i.e., $\langle \hat{v}^2 \rangle = \int \mathcal{D}v e^{iS/\hbar} v^2$, where S is the action and $\mathcal{D}v$ is a path integral measure over all possible configurations of v). If we take the expectation value of Eq. (11a), we almost have a closed system of differential equations relating w and $\langle \hat{v} \rangle$. The $\langle \hat{v}^2 \rangle$ term from Eq. (11b) and the $\langle \hat{v}^3 \rangle$ term from Eq. (11a) prevent the equations from being closed. Applying the operation $x^2\partial^2/\partial x^2$ to the operator $\hat{v}^2$ and using Eq. (11a) yields

$$x^2(\hat{v}^2)'' = 2\hat{v}^2(\hat{v}^2 - 1 - w) + 2x^2(\hat{v}')^2 \tag{12}$$

If we take the expectation of the above equation with respect to fluctuations in the ansatz function operator $\hat{v}^2$ and combine this with Eq. (11b) we almost have a closed system for determining w and $\hat{v}^2$ except for the $\langle(\hat{v}')^2\rangle$ term which comes from the last term on the right-hand side of Eq. (12). Continuing in this way, one can obtain an infinite set of equations for various powers of the ansatz function operator (i.e., $\hat{v}^n$). These higher order equations never close. To deal with this problem we follow Heisenberg, and make some assumption that effectively cuts off the system of equations at some finite order. By taking the expectation of Eq. (12) and further by assuming that

$$\langle(\hat{v}')^2\rangle = \frac{\langle \hat{v}^2 \rangle - v_0^2}{x^2} \tag{13}$$

we arrive at a closed system of equations from Eqs. (11b), (12):

$$x^2\langle \hat{v}^2 \rangle'' = 2\langle \hat{v}^2 \rangle^2 - 2\langle \hat{v}^2 \rangle w - 2v_0^2 \tag{14a}$$

$$x^2w'' = 6w\langle \hat{v}^2 \rangle \tag{14b}$$

By making the assumption of Eq. (13) we have simplified Eqs. (11b) and (12) to the closed system given by Eqs. (14a) and (14b). It is straightforward to show that in the limit $x \to \infty$

$$\langle \hat{v}^2 \rangle = v_0^2 + \frac{a}{x^\alpha} \tag{15a}$$

$$w = \frac{b}{x^\alpha} \tag{15b}$$

solves Eqs. (14a) and (14b) provided that $v_0^2 = 1$, $b = -a$, and $\alpha = 2, -3$. In order for $\langle \hat{v}^2 \rangle$ and w to have acceptable behavior at $x \to \infty$ we take the $\alpha = 2$ solution. Substituting the above expressions for $\langle \hat{v}^2 \rangle$ with $v_0 = +1$ and w back into Eq. (11a), and assuming that $\langle \hat{v}^3 \rangle = \langle \hat{v} \rangle \langle \hat{v}^2 \rangle$ gives the following equations for $\langle \hat{v} \rangle$ in the $x \to \infty$ limit

$$x^2 \langle \hat{v} \rangle'' = \langle \hat{v} \rangle (\langle \hat{v}^2 \rangle - 1) = \langle \hat{v} \rangle \frac{a}{x^2} \tag{16}$$

we find that Eq. (16) is solved in the $x \to \infty$ limit by

$$\langle \hat{v} \rangle = \pm \left(1 + \frac{a}{6x^2} \right) \tag{17}$$

Equation (17) together with Eqs. (15a) and (15b) provide information on the behavior of the "classical" ansatz function w and the "quantum" ansatz function v via $\langle \hat{v} \rangle$ and $\langle \hat{v}^2 \rangle$. The main point of interest is that after applying the Heisenberg-like quantization procedure to the classical solution of Fig. 1, the infinite increase of the ansatz function w has changed to an acceptable asymptotic behavior (i.e., one that leads to a finite field energy). By replacing the v^2 and $(v')^2$ terms in Eq. (9) with $\langle \hat{v}^2 \rangle$ and $(\langle \hat{v} \rangle')^2$ from Eqs. (15a) and (17), respectively, and also using w from eq. (15b) we find that the field energy density of the quantized "bunker" solution takes the form

$$\mathcal{E} \propto \frac{a^2}{x^8} \tag{18}$$

in the limit in which quantum fluctuations become important (i.e., for non-Abelian theories which exhibit asymptotic freedom this means in the low-energy or $x \to \infty$ range) the energy density goes from the form given in Eq. (9) to that given in Eq. (18). This can be seen to give a finite field energy. In the high-energy or short-distance regime we assume that the fields approach the classical configuration of Fig. 1 due to asymptotic freedom. This classical configuration is well behaved at $x = 0$, but would yield an infinite field energy due to its divergence as $x \to \infty$. In the long-distance or low-energy limit the energy density should go over into the form given in Eq. (18), which would then result in a finite field energy for this configuration, since the integral of Eq. (18) over the large-x region, where it is valid, would give a finite field energy.

Finally, by using w, $\langle\hat{v}\rangle$, and $\langle\hat{v}^2\rangle$ in the expressions for **E** and **H** given in Eq. (7) we find that the radial fields (E_r, H_r) go like a/r^4 while the angular fields ($E_{\theta,\phi}$, $H_{\theta,\phi}$) go like a/r^3. Thus the quantization procedure outlined above modifies the undesirable long distance behavior of the "electric" and "magnetic" fields as well as the energy density.

4. DISCUSSION

In this paper we reviewed a certain classical field configuration for an *SU*(3) gauge theory. Near the origin the field configurations are finite, but as $r \to \infty$ the fields diverge (see Figs. 1 and 2). This increasing field strength leads to a classical type of confinement in that a test particle placed in the background field of this solution cannot escape to ∞. Unfortunately, this diverging of the field as $r \to \infty$ also leads to this configuration having an infinite field energy. Previously it was suggested that quantum effects might soften or eliminate this bad long-distance behavior. By applying a method similar to that Heisenberg used in quantizing the nonlinear Dirac equation we find that the long-distance behavior is changed so as to give finite field energy. At short distances the fields should approach the classical configuration of Fig. 1 from the asymptotic freedom of the *SU*(3) gauge theory. This classical solution has the good features of not being divergent at $r = 0$ and in some limited region around $r = 0$ the fields increase in a way similiar to that found in some phenomenological models of confinement. At long distances the fields should approach the configuration given by Eqs. (15a), (15b), and (17) where the quantum effects have eliminated the divergence of the fields and field energy density as $r \to \infty$.

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